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Probability Distributions

G.N.Dar

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0.1 Binomial Distribution (discret dist.)

In *random experiments* we have seen random variables X both discrete and continous in nature. Both the variables take finite and infinite number of valuse x and corresponfing to each value is associated a Probability or Probability function $f(x) = P(X = x)$. This $f(x)$ verses x defines a distribution. There are number of distributions both discrete as well as continous corresponfing to the discrete random variable as well as continous random variable.

When the outcome of random experiment (mutually exclusive) is just win or loss, pass or fail say head or tail; both equally probable. And if we go for n independent trails or random experiments till getting some desirable outcome. This will follow a distribution called as *Binomial Distribution B(n,x)*.

Example: Tossing two coinst and let X represent the number of heads, we know

$$X = 2, 1, 0$$

with respective probabilities as ($p = \frac{1}{2}$)

$$P(X = x) = f(x) = \frac{1}{4}, \frac{1}{2}, \frac{1}{2}$$

total Probability

$$P(2) + P(1) + P(0) = 1$$

this can be written as keeping in view the possible permetation combination say for 1 head $\{H, T\}$ or $\{T, H\}$

$$TotalProb =^n C_r p^n q^{n-r}$$

=

$$p^2q^0 + 2.p^1q^1 + p^0q^2 = \frac{1}{4} + 2.\frac{1}{4} + \frac{1}{2} = 1$$

Derivation of Binomial Distribution: Let X a discret random variable represent the number of trial (tossing a n coin together or one coin n times) when an even say A (for example head) ocurs with a probability $P(X = A) = P(A)$; it can be

$$X = \{0, 1, 2, 3, 4, \cdots, n\}$$

And let the complemt of event A is $B = A^c$ There is possibility as if r times event A occurs then $n - r$ event B occurs which can be shown as (trials are independent)

$$AAAAAAAAA \cdots A_r.BBBBBBBB \cdots B_{n-r}$$

Recall if

$$P(A) = p, \qquad P(B) = P(A^c) = 1 - P(A) = 1 - p = q$$

we can write the probability of above case as as events are independent

$$pppppp \cdots p_r.qqqqq \cdots q_{n-r}$$

that is just one case . All the possible outcomes can be written as $f(x)$ is the probability distribution function which should be just 1

$$f(x) = P(X = x) =^n c_r p^r q^{n-r} =^n c_r p^r (1 - p)^{n-r}$$

for example for $n = 1$ (single coin)

$$f(x) =^1 c_r p^r (1 - p)^{n-r}$$

$$f(x) =^1 c_r p^r (1 - p)^{1-r}$$

r can take only two values either success say $r = 0$ else failure say $r = 1$ therefore

$$f(x) = \sum_{r=0}^{r=1} =^n c_r p^r (1 - p)^{n-r}$$

$$^1c_0 p^0 (1 - p)^{1-0} +^1 c_1 p^1 (1 - p)^{1-1} = (1 - p) + p = 1$$

Similarly for two coin tossing we have the following possibility

$$f(x) = \sum_{r=0}^{r=2} =^2 c_r p^r (1 - p)^{2-r}$$

$$f(x) =^2 c_0 p^0 (1 - p)^{2-0} +^2 c_1 p^1 (1 - p)^{2-1} +^2 c_2 p^2 (1 - p)^{2-2} = \frac{1}{4} + 2\frac{1}{4} + 1.\frac{1}{4} = 1$$

We can plot this for two coin toss ans see how Binomial Distribution looks like for the case of of a random variable X representing sum of two fair dice throw. when

$$X = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$$

ther are $n = 11$ dinstict cases label them as $r = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$ And keeping in veiw the 36 number of possible casees and permetation and combination we for example 12 occurs only once while as 11 occurs twice by a combination of $(5, 6)$ and $(6, 5)$ therfore accordigly probabilities are shown as

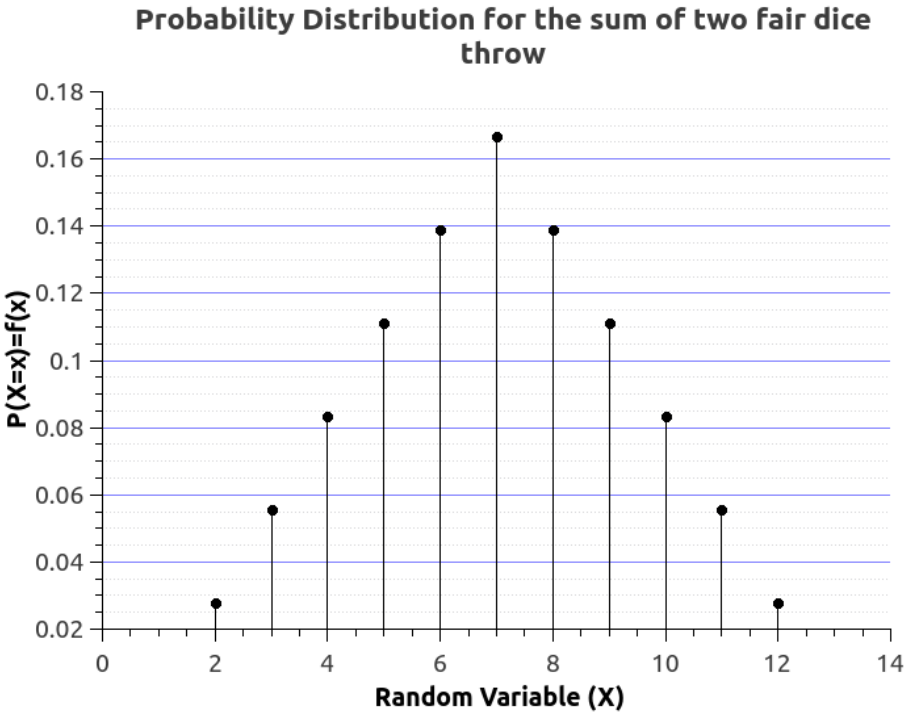

Figure 1: Discrete Distributions

$$f(x) = \frac{1}{36}, \frac{2}{36}, \frac{3}{36}, \frac{4}{36}, \frac{5}{36}, \frac{6}{36}\frac{5}{36}\frac{4}{36}\frac{3}{36}\frac{2}{36}\frac{1}{36}$$

OR

$f(x) = 0.0278, 0.0556, 0.0834, 0.111, 0.139, 0.166666668, 0.1390.111, 0.0834, 0.0556, 0.0278$

$$^{n}c_0 p^r q^{n-r} =^{n} c_r \frac{1}{36}^{r} (1 - \frac{1}{36})^{n-r})$$

$$=^{12} c_0 \frac{1}{36}^{0} (1 - \frac{1}{36})^{12-0} +^{12} c_1 \frac{1}{36}^{1} (1 - \frac{1}{36})^{12-1} +^{12} c_2 \frac{1}{36}^{2} (1 - \frac{1}{36})^{12-2} + \cdots +^{12} c_{12} \frac{1}{36}^{1} 2(1 - \frac{1}{36})^{12-12} = 1$$

0.2 Parameters of Binomial Distribution

Expectation Value:

We know

$$f(x) =^{n} c_x p^x q^{n-x}$$

And

$$E(X) = \sum_{x=0}^{n} x.{}^{n}c_{x} p^{x} q^{n-x}$$

$$= \sum_{x=0}^{n} x.\frac{n!}{x!(n-x)!} p^{x}(1-p)^{n-x} = \sum_{x=1}^{n} x.\frac{n!}{(x-1)!(n-x)!} p^{x}(1-p)^{n-x}$$

when

$$y = x - 1$$

we get

$$x = y + 1$$

And at the same time we have some changes in index n as when

$$m = n - 1$$

we get

$$n = m + 1$$

and when

$$n = m + 1$$

when

$$x = 1 \to n$$

we have for y as

$$y = 0 \to (n-1) = m$$

$$n = m + 1$$

Therfore

$$E(X) = \sum_{y=0}^{m} \frac{(m+1)!}{y!(m-y)!} p^{y+1}(1-p)^{m-y}$$

$$= (m+1)p.\sum_{y=0}^{m} \frac{m!}{y!(m-y)!} p^{y}(1-p)^{m-y}$$

$$= np.\sum_{y=0}^{m} \frac{m!}{y!(m-y)!} p^{y}(q)^{m-y} = np.1 = np$$

Hence for Binomial Distribution is given as

$$E(X) = np$$

For 100 coin toss or tossing a coin 100 times $E(X) = np = 100.\frac{1}{2} = 50$, we expect 50 heads and 50 tails.

Example: Toss three coins and let X represent number of head. Find $f(x), F(x)$ and plot the Binomial distribution $f(x)$ verses $X = x$

Variance $\sigma^2(x)$:

$$\sigma^2(x) = \nu_2 = E(X^2) - (E(X))^2 = E(X(X-1)) + E(X) - (E(X))^2$$

In order to get the variance we just need to evalute $E(X(X-1))$ and rest is just expectation value $E(X)$ so we can write analogically

$$E(X(X-1)) = \sum_{x=0}^{n} x(x-1)f(x) = \sum_{x=0}^{n} x(x-1)^n c_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1)\frac{n!}{(n-x)!x!}p^x(1-p)^{n-x}$$

$$= \sum_{x=2}^{n} \frac{n!}{(n-x)!(x-2)!}p^x(1-p)^{n-x}$$

$$E(X(X-1)) = n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(n-x)!(x-2)!}p^{x-2}(1-p)^{n-x}$$

which can be shown to be equal to as per previous little play

$$(x-2=y) of mathematics$$

$$E(X(X-1)) = n(n-1)p^2 \sum_{y=0}^{m} \frac{m!}{(m-y)!y!}p^y(1-p)^{m-y} = n(n-1)p^2.1 = n(n-1)p^2$$

Therfore variance is give

$$\sigma^2(x) = \nu_2 = E(X^2) - E(X)^2 = E(X(X-1)) + E(X) - E(X)^2 = n(n-1)p^2 + np - (np)^2 = npq$$

OR

$$\sigma^2(x) = npq$$

Example: Do some examples yourself and plot a Binomial Distribution on them.

Example:
An automatic camera records the number of cars running a red light at an intersection (that is, the cars were going through when the red light was against the car). Analysis of the data shows that on average 15% of light changes record a car running a red light. Assume that the data has a binomial distribution. What is the probability that in 20 light changes there will be exactly three (3) cars running a red light? Write out the key statistics from the information given: $p = 0.15, n = 20, X = 3$ Apply the formula, substituting these values:$^n c_x p^x q^{n-x} =^{20} c_3(0.15)^3(1-0.15)^{20-3} = 0.24$

That is, the probability that in 20 light changes there will be three (3) cars running a red light is $0.24 = (24\%)$.

0.3 Problems on Binomial Distribution (Home Work)

1. Executives in the New Zealand Forestry Industry claim that only 5% of all old sawmills sites contain soil residuals of dioxin (an additive previously used for anti-sap-stain treatment in wood) higher than the recommended level. If Environment Canterbury randomly selects 20 old saw mill sites for inspection, assuming that the executive claim is correct: a) Calculate the probability that less than 1 site exceeds the recommended level of dioxin. b) Calculate the probability that less than or equal to 1 site exceed the recommended level of dioxin. c) Calculate the probability that at most (i.e., maximum of) 2 sites exceed the recommended level of dioxin. 2 Inland Revenue audits 5% of all companies every year. The companies selected for auditing in any one year are independent of the previous year's selection. a) What is the probability that the company 'Ross Waste Disposal' will be selected for auditing exactly twice in the next 5 years?

b) What is the probability that the company will be audited exactly twice in the next 2 years? c) What is the exact probability that this company will be audited at least once in the next 4 years? 3 The probability that a driver must stop at any one traffic light coming to Lincoln University is 0.2. There are 15 sets of traffic lights on the journey. a) What is the probability that a student must stop at exactly 2 of the 15 sets of traffic lights? b) What is the probability that a student will be stopped at 1 or more of the 15 sets of traffic lights?