Numerical Analysis

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Interpolations

It is a study of algorithms using numerical approximations for mathematical analysis.In this course we will see number of methods for fitting a given data. Each method involves some error and looking for improvement gives birth to a new techniques/methods.

Motivations

To find a *polynomial* $f(x)$ or $p_n(x)$ for a given data set (x_i, y_i) in such a way that at given values of $x, f(x_i) = y_i = P_n(x)$
$i = 0, 1, 2, \cdots$ ....n
For example

$$f(x_0) = y_0, f(x_1) = y_1, f(x_2) = y_2, \cdots f(x_n) = y_n$$

Missing term estimation

1. This polynomial is used to find the missing terms in the data set or pridict the expected values of y_i for x_i

2. If the missing term lies in between the given data set say x_0 to x_n The process is known as interpolation.

3. If the missing term lies outside the given data set, the process is known as Extrapolation

4. The process involves to find the coefficients of fitted polynomial using given data set.

As interpolation is the techniques of connecting points accordingly we have

- linear fit (using two data point)
- parabolic fit (using three data point)
- cubic (using four data point)
- quad (using five data point)
- and n^{th} degree polynomial using $n+1$ data points

Methods of finding a function (Polynomial)

Direct Method

For a given data set of $(n+1)$ points $x_i, y_i, i = 0, 1, 2, \cdots n$

Let

$$P_n(x) = f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + .. + a_nx^n$$

is the required polynomial setisfying the given data such that $P_n(x_i) = y_i$ for each i , we can write for example

$$P_n(x_0) = f(x_0) = a_0 + a_1x_0 + a_2x_0^2 + a_3x_0^3 + .. + a_nx_0^n$$

$$P_n(x_1) = f(x_1) = a_0 + a_1x_1 + a_2x_1^2 + a_3x_1^3 + .. + a_nx_1^n$$

$$P_n(x_2) = f(x_2) = a_0 + a_1x_2 + a_2x_2^2 + a_3x_2^3 + .. + a_nx_2^n$$

and

$$P_n(x_n) = f(x_n) = a_0 + a_1x_n + a_2x_n^2 + a_3x_n^3 + .. + a_nx_n^n$$

this can be solved for the coefficents $a_0, a_1, a_2, \cdots a_n$ and the polynoial is obtained.

We can write above set of equations in a matrix form and write the value of given parameters and solve for coefficents

$$\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix} \times \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

Example

Let x_i and y_i is given as $x_i = 1, 2, 3$ and $y_i = 1, 4, 9$. Fit a polynomial through direct method. Write your expression and find $p_n(x)$ at $x = 4$

Langrange Interpolation

Aim is to determine the polynomial for a data set which is not equally spaced. Let x_i amd f_i , where $i = 0, 1, 2, 3, \cdots n$ is a given data set with arbitary spaced.

Langrange idea is to multiply each f_j by a *polynomial* that is just 1 at x_j and just 0 elsewhere (other than x_j . Then take sum of these $n + 1$ *polynomials* to get the *unique* interpolation formula of degree n or else. Let us work it out in steps.

Linear Interpolation

Consider two point interpolation through (x_0, y_o) and (x_1, y_1). Then the *Linear Langrange interpolation* is

$$P_1 = L_0(x)y_0 + L_1(x)y_1$$

where $L_0(x)$ and $L_1(x)$ is a polynoial given as
$L_0(x) = 1$at $x = x_0$ and L$_0(x) = 0$ eles where and
$L_1(x) = 1$ *at* x= x$_1$ and L$_1(x) = 0$ eles where

therefore we write the expressions for polynomials as per rule as

$$L_0(x) = \frac{x - x_1}{x_0 - x_1}$$

$$L_1(x) = \frac{x - x_0}{x_1 - x_0}$$

with this our solution or Linear interpolation formula is

$$P_1(x) = (\frac{x - x_0}{x_0 - x_1})y_0 + (\frac{x - x_0}{x_1 - x_0})y_1$$

Assignment1: Replace x_0 by x_1 and x_1 by x_2 and that above equation $P_1(x)$ is a strieght line given as (in terms of x and y)

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

Which is like

$$y = mx + c$$

Quadratic Interpolation

Second degree interpolation takes care of three points instead of two as in linear interpolation. for example consider the data string as $(x_0, y_0), (x_1, y_1), (x_2, y_2)$. Then $P_2(x)$ is obtained using Langrange Interpolation rule as

$$P_2(x) = L_0(x)y_0 + L_1(x)y_1 + L_2(x)y_2$$

such that each $L_0(x), L_1(x)$ and $\mathrm{L}_2(x)$ is a polynoial given respectively as

$L_0(x) = 1$ or o at $x = x_0$ and elesewhere x_0
$L_1(x) = 1$ or o at $x = x_1$ and elesewhere x_1
$L_2(x) = 1$ or o at $x = x_2$ and elesewhere x_2

The only choice to have the above is as

$$L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{l_0(x)}{l_0(x_0)}$$

$$L_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{l_1(x)}{l_0(x_1)}$$

$$L_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{l_2(x)}{l_0(x_2)}$$

In general

$$L_k(x_j) = \frac{(x_j - x_l)(x_j - x_m)}{(x_j - x_l)(x_j - x_m)}$$

Note that $L_k(x_j) = 0$ if $k \neq j$ otherwise 1

Quadratic Interpolation:

Similar to above interpolation here we take 04 points together and develop a polynoial of degree 3 as per *Langrange rule* as

$$P_3(x) = L_0(x)y_0 + L_1(x)y_1 + L_2(x)y_2 + L_3(x)y_3$$

with the parameter defined as

$$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{l_0(x)}{l_0(x_0)}$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{l_1(x)}{l_1(x_1)}$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{l_2(x)}{l_2(x_2)}$$

$$L_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{l_3(x)}{l_3(x_3)}$$

In short we can write

$$P_3(x) = \sum_{k=0}^{3} L_k(x)y_k = L_0(x)y_0 + L_1(x)y_1 + L_2(x)y_2 + L_3(x)y_3$$

The general n^{th} degree polynoial is given as

$$P_n(x) = \sum_{k=0}^{n} L_k(x)y_k = L_0(x)y_0 + L_1(x)y_1 + L_2(x)y_2 + \cdots + L_n(x)y_n$$

Or

$$P_n(x) = \sum_{k=0}^{n} \frac{l_k(x)}{l_k(x_k)} y_k$$

where $L_k(x) = \frac{l_k(x)}{l_k(x_k)} y_k$. Such that $L_k(x_k) = 1$ and 0 at other nodes. Note down for polynoial of degree n

$$l_0(x) = (x-x_1)(x-x_2)(x-x_3)\cdots(x-x_n)$$

$$l_1(x) = (x-x_0)(x-x_2)(x-x_3)\cdots(x-x_n)$$

$$l_2(x) = (x - x_0)(x - x_1)(x - x_3)\cdots(x - x_n)$$

$$l_3(x) = (x - x_0)(x - x_1)(x - x_2)\cdots(x - x_n)$$

$$l_n(x) = (x - x_0)(x - x_1)(x - x_2)\cdots(x - x_{n-1})$$

Also note that or check

$$P_n(x_0) = y_0$$

$$P_n(x_1) = y_1$$

$$P_n(x_n) = y_n$$

Assignment

For the given table find the Langrange interpolation formula

$x_k \rightarrow$	1	0	2	3
$y_k \rightarrow$	-1	2	10	35

Show that

$$P_3(x) = \frac{5}{3}x^3 - \frac{4}{3}x^2 + 2$$

Verify $P_3(x_0) = y_0 = -1$

Issues with Langrange interpolation formula

- Amount of computations required is large. It involves $2(n+1)$ multiple / divisional and $(2n + 1)$ additional operations . Where as power polynoial $a_0+a_1x+\cdots+a_nx^n$ involves only n multiple and n additional operations

- Interpolation for additional value of x requires same amount of effort as first value. It does not take it from previous calculations and starts from begning.
- If in the middle of calculations, number of interpolation points are changed . Previous results are not used.
- And error is large

Newtons Method Here in this method we are able to increase order of polynoial whereas in Langrange entirely a new operation has to start up from the begning. This method is used for equally spaced arguments $(x_{i+1} - x_i = h)$ say is always uniform.

Since Interpolation is needed to estimate the missing terms. Missing terms can be in the begning , end or in the middle of given data. for example through census i know the population from 1981 till 2011. If I am interested to know before 1981 or in the middle say 2004 or at the end around say 2014 . Accordingly we have three types of Newtons interpolation methods defined through difference tables, taken as

- Forward differences
  Backward differences
  Central differences

Interpolations defined through differences depend on difference tables. Before going to Newtons Methods we will learn first of difference tables. difference of dependent variable is involed say $y_i - y_[i + 1$ as per the tables .

0.1 Forward Difference Table

x	**y**	**Δ (first difference)**	**Δ^2 (second difference)**	**Δ^3(third difference)**
x_0	y_0			
x_1	y_1	$\Delta y_0 = y_1 - y_0$		
x_2	y_2	$\Delta y_1 = y_2 - y_1$	$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$	
x_3	y_3	$\Delta y_2 = y_3 - y_2$	$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$	$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$

Note: Symbolic representation y_0 which is subtracted.

0.2 Backward Difference Table

x	**y**	**∇ (first difference)**	**∇^2 (second difference)**	**∇^3(third difference)**
$\mathbf{x}_0$	$\mathbf{y}_0$			
$\mathbf{x}_1$	$\mathbf{y}_1$	$\nabla y_1 = y_1 - y_0$		
$\mathbf{x}_2$	$\mathbf{y}_2$	$\nabla y_2 = y_2 - y_1$	$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$	
$\mathbf{x}_3$	$\mathbf{y}_3$	$\nabla y_3 = y_3 - y_2$	$\nabla^2 y_3 = \nabla y_3 - \nabla y_2$	$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2$

Note: Symbolic representation y_1, y_2, y_3 from which subtracted

Assignment: choose your data from x_0, y_0 to x_6, y_6. Derive the result for

$\Delta y_0, \Delta y_4, \Delta^2 y_0, \Delta y_4, \Delta y_5, \Delta^4 y_0,$

And for

$\nabla y_1, \nabla y_4, \nabla^2 y_2, \nabla^3 y_3$

Note: See forward difference table end at y_0 used for forward difference interpolation formula. Backward difference table ends at y_n , the last data point used for Backward difference interpolation formula.

1 Newtons forward interpolation formula (NFIF)

If $x_0, x_1, x_2, ..., x_k and$ $\mathbf{y}_0, y_1, y_2, ...y_k$ are give set of observations with arguments equally spaced.The NFIF is given as

$$P_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2}{3!}\Delta^3 y_0 + \cdots \frac{p(p-1)(p-2)/cdots(p-(n-1)}{n!}\Delta^n y_0$$

where $p = \frac{x - x_0}{h}$, h a uniform spacing between arguments.
Proof: Assume n^{th} degree polynoial for a given data as

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1)(x - x_2)\cdots(x - x_{n-1})$$

at

x=x$_0$, $P_n(x_0) = a_0 = y_0$

as it must be
at

x=x$_1$, $P_n(x_1) = y_1 = a_0 + a_1(x_1 - x_0) = y_0 + a_1 * h$

from this we get

$\mathbf{a}_1 = \frac{(y_1 - y_0)}{h} = \frac{\Delta y_0}{h}$

Similarly at

x=x$_2$

$\mathbf{P}_n(x_2) = y_2 = a_0 + a_1(x_2 - x)) + a_2(x_2 - x_0)(x_2 - x_1) = y_0 + 2_0 + 2h^2 a_2$

$\mathbf{a}_2 = \frac{\Delta^2 y_0}{2h^2}$

at

$$x = x_3$$

$$P_n(x_3) = y_3$$

gives expression for a_3 as

$$a_3 = \frac{\Delta^3 y_0}{3! h^3}$$

and soon

therefore with all these coeffients so evaluated we end up with an expresion involing various difference operators and y_0 called as Newtons forward interpolation polynoial as

$$P_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots + \frac{p(p-1)\cdots(p-(n-1))}{n!}\Delta^n y_0$$

where $p = \frac{x - x_0}{h}$
The above formula is also known as Newton -Gregory forward polynoial. This is like a Binomial.

1.1 Example : Home work

Develop forward difference table and hence a NFIF for

x	y	Δ	Δ^2	Δ^3
2	**-7**			
4	**-3**			
6	**6**			

Write NFIF and substitute the valuse from table and find the polynomial.

2 Newtons Backward interpolation formula

is Similar to the forward with little difference as is given below

$$P_n(x) = y_n + (x - x_n)\frac{\nabla y_n}{h} + \frac{1}{2!}(x - x_n)(x - x_{n-1})\frac{\nabla^2 y_n}{h^2} + \frac{1}{n!}(x - x_n)(x - x_{n-1})...(x - x_1)\frac{\nabla^n y_n}{h^n}$$

Derive the above mentiond formula using the assumption

$$P_n(x) = a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) + \cdots + a_n(x - x_n)(x - x_{n-1}) \cdots (x - x_1)$$

At given valuse , polynomial has to pass through given points a required condition for interpolation. Derive NBIF after determing all the coefficents (Home Work).

On putting $(x - x_n) = p * h$ **and determiing all the terms like** $(x - x_n), (x - x_{n-1} = (p + 2) * h$ **and** $(x - x_{n-2}) = (p + 3) * h$ **and soon . Wecan write the NBIF as**

$$P_n(x) = y_n + p\nabla y_n + \frac{p(p + 1)}{2!}\nabla^2 y_n + \cdots + \frac{p(p + 1)(p + 2).....(p + n - 1)}{n!}\nabla^n y_n$$

Assignment

Derive above polynoial. And for the given table, estimate $P_n(7.5)$

x	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
y	**1**	**8**	**27**	**64**	**125**	**216**	**343**	**512**

Note: As the point of estimate lies near end so Backward interpolation will be the most suitable. Please evaulate the result by using Newtons forward interpolation formula also.

Answer: $P_n(7.5) = 421.875$, **Since from data it is clear that the table is showing cubic behavior which cab be checked also by calulating** $(7.5)^3$

2.1 Home work

population of town is given below. Estismate the population in 1955 **and** 1985

year	**1951**	**1961**	**1971**	**1981**	**1991**
population	**46**	**66**	**81**	**91**	**101**

Choose your interpolation as per the question i,e where will you use forward and where will you use Backward interpolation formula.

Answer: for $1955 = 54.85$ **And for** $1985 = 96.84$

3 Central Differences

The central difference operator δ **is defined as**

$$y_1 - y_0 = \delta y_{\frac{1}{2}}$$

,

$$y_2 - y_1 = \delta y_{\frac{3}{2}}$$

. . . .

$$y_n - y_{n-1} = \delta y_{n-\frac{1}{2}}$$

similar higher order central difference can be defined

3.1 Central Difference Table

x	y	δ (first difference	δ^2 (sec.diff)	δ^3 (third diff)	δ^4 (fourth diff)
x_0	y_0				
x_1	y_1	$\delta y_{\frac{1}{2}}$			
x_2	y_2	$\delta y_{\frac{3}{2}}$	$\delta^2 y_1$		
x_3	y_3	$\delta y_{\frac{5}{2}}$	$\delta^2 y_2$	$\delta^3 y_{\frac{3}{2}}$	
x_4	y_4	$\delta y_{\frac{7}{2}}$	$\delta^2 y_3$	$\delta^3 y_{\frac{5}{2}}$	$\delta^4 y_2$

The rule is as

$$\delta^n y_{r-\frac{1}{2}} = \delta^{n-1} y_r - \delta^{n-1} y_{r-1}$$

for odd n **and** $r = 1, 2, 3....$ **so we have with this** $\delta, \delta^2, \delta^3, \delta^4, ...$

$$\delta^n y_r = \delta^{n-1} y_{r+\frac{1}{2}} - \delta^{n-1} y_{r-\frac{1}{2}}$$

for even n **and** $r = 1, 2, 3....$ **so we have with this** $\delta^2, \delta^4, \delta^6, \delta^8, ...$

3.2 Home work

Derive Gauss central difference interpolation formula.

4 Symbolic Relations between the operators

In addition to forward difference operator Δ, Backward difference operator ∇ and central difference operator δ, we have shift operator $E^{\pm}$ which raises or lowers the change in unit 1 or h depending on the sign.

4.1 Shift operator E

$$Ey_x = y_{x+h}$$

Similarly

$$E^2 y_x = E(Ey_x) = E(y_{x+h}) = y_{x+2h}$$
$$E^3 y_x = E^2(Ey_x) = E(y_{x+2h}) = y_{x+3h}$$

$\vdots$

$$E^n y_x = y_{x+nh}$$

4.2 Relations

we know

$$\Delta y_x = y_{x+h} - y_x = Ey_x - y_x$$
$$\Delta = E - 1$$

$$1 + \Delta = E$$

Similar with Backward difference operator ∇ we have

$$\nabla y_x = y_x - y_{x-h} = y_x - E^{-1} y_x$$
$$E^{-1} = 1 + \nabla$$

Similary we can prove

$$E\nabla y_x = \Delta y_x$$

or

$$E\nabla = \Delta$$

And we can also prove

$$E\nabla = \nabla E = \Delta$$

Verify them all.

Similar prove with centra difference operator

$$\delta y_x = y_{x+\frac{h}{2}} - y_{x-\frac{h}{2}} = [E^{\frac{1}{2}} - E^{\frac{-1}{2}}]y_x$$

$$\delta = E^{\frac{1}{2}} - E^{\frac{-1}{2}}$$

we can also prove

$$\Delta = \delta E^{\frac{1}{2}}$$

we can also show

$$Ef(x) = f(x+h) = e^{hD}f(x)$$

and using Taylor expansion we can define as

$$E = e^{hD}$$

Note Prove all these relations.

4.3 Assignment

1. See how difference table is used for detection of errors.

2. Study Gauss interpolation polynomial.

3. Missing term etimations using

a) difference operators b) difference table

Let us see this example

x	45	50	55	60	65
y	3	...	2	...	2.4

Solution Given only 3 **values so** $(n+1) = 3$ **and only second difference will be constant and third will be zero.**

Here x$_0, x_1, x_2, x_3, x_4$ **is given but only** y_0, y_2, y_4 **is given, two missing terms** y_1 **and** y_3 **needs to be estimated**

we know

$$\Delta^3 y_0 = 0$$

but

$$\Delta = E - 1$$

therfore

$$(E-1)^3 y_0 = 1 = (E^3 - 3E^2 + 3E - 1)y_0$$

$$y_3 - 3y_2 + 3y_1 - y_0 = 0 \tag{5}$$

similarly

$$\Delta^3 y_1 = 0$$

gives

$$y_4 - 3y_3 + 3y_2 - y_1 = 0 \tag{6}$$

Solve for given valuse and estimate the required missing terms with the help of above two equaitons and the given table.

Using Difference Table right