Random Variables

G.N.Dar

May 25, 2020

1 Random Variables

1. Toss a coin twice and let X represent the number of head, we have

   X: 2 1 1 0

2. In the same experiment if X represent squre of heads we have

   X: 4 1 1 0

3. And if X represent the difference of number of heads and tails we have

   X: 2 0 0 -2

In all these cases depending on the definition we see X a varaible takes a finite discrete values. Sometimes this varaible can take countably infinite number of values or Sometimes it can take infinitely uncountable values. The variable X is defined as *Random Varaible.* The former when it takes finite number of values is called as *Discrete Random Variable* and the latter when it takes uncountable infinite number of values is called as *Continous Random Variable* .

Associated to each discrete value of X say $X = x_k$ is a probability as $P(X = x_k) = f(x_k)$ called as probability function.

Remeber $i) f(x_k) \geq 0$

$ii)$ $\sum_k f(x_k) = 1$ where $k = 1, 2, 3.....n$ is called as probability Distribution Function.

The function associated to each Continous discrete random varaible is called as probability density function. For

Example:

Probability function in experiment 1 above

$$P(X = 0) = P(TT) = f(x) = \frac{1}{4}$$

$$P(X = 1) = P(HT \cup TH) = f(x) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$P(X = 2) = P(HH) = f(x) = \frac{1}{4}$$

x	0	1	2
f(x)	0.25	0.5	0.25

$$\sum_{i=1}^{3} f(x) = 1$$

We define Distribution Funciton $F(x) as$

$$F(x) = P(X \leq x), x : -\infty \leq x \leq +\infty$$

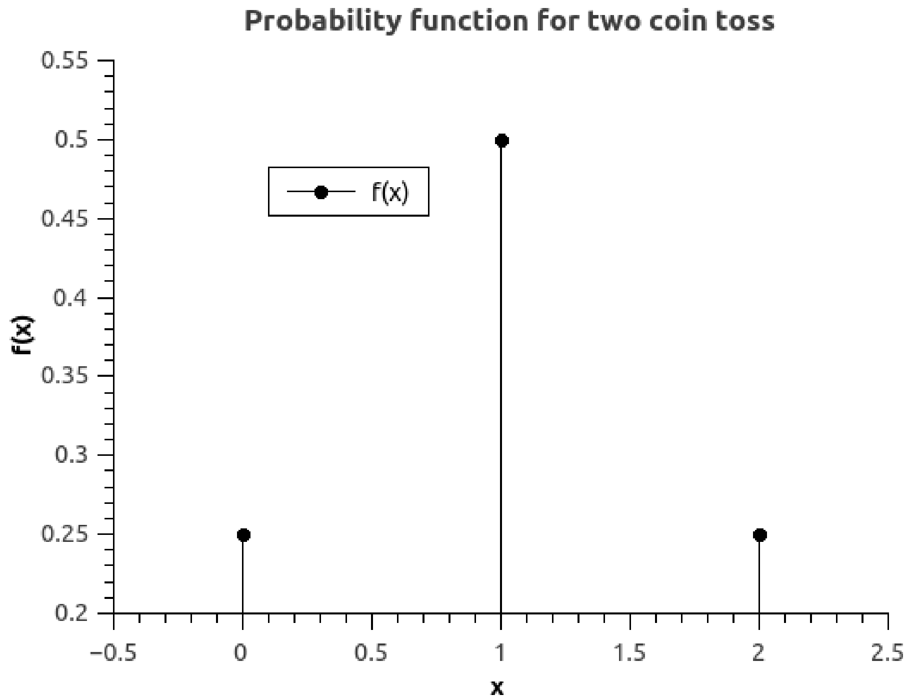

Figure 1: Probability function

Similar for a single dice we have probability functions as

$$P(X=1) = f(1) = \frac{1}{6}, P(X=2) = f(2) = \frac{1}{6}, P(X=3) = f(3) = \frac{1}{6}, P(X=4) = f(4) = \frac{1}{6}$$

$$P(X=5) = f(5) = \frac{1}{6}, P(X=6) = f(6) = \frac{1}{6}$$

and the Distribution functions will be as

$$F(x) = P(X \leq x), x : 1 \to 6$$

As a note to mark that $P(X \geq 5) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6}$

Example: For two fair dice throw let X represent the sum of faces . Write Probability function $f(x)$ and Distribution funciton $F(x)$.

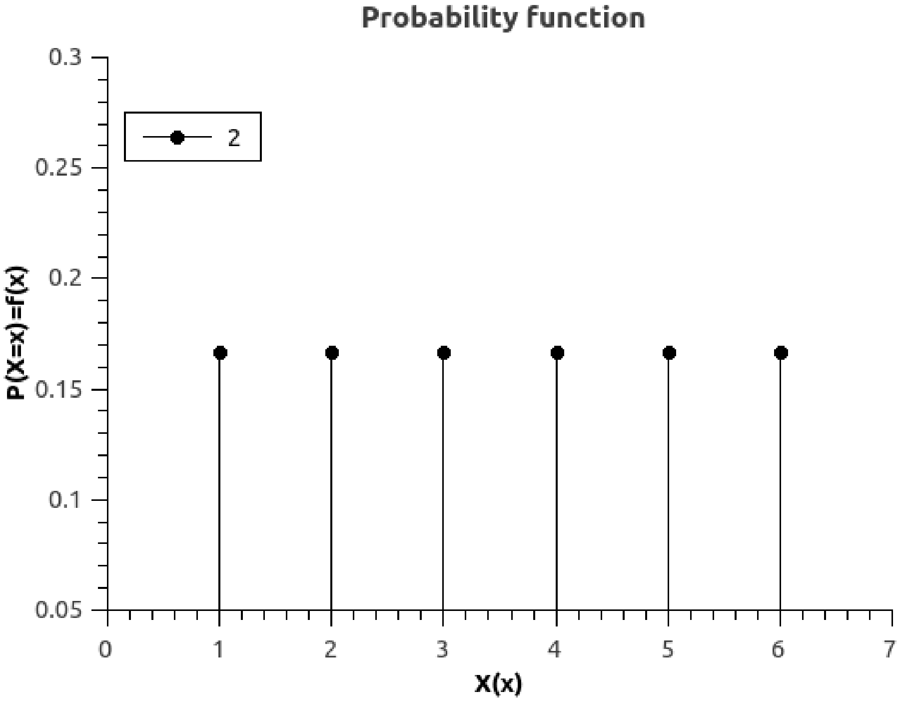

Figure 2: Probability function

Solution:

$$X = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$$

$$f(x) = \frac{1}{36}, \frac{2}{36}, \frac{3}{36}, \frac{4}{36}, \frac{5}{36}, \frac{6}{36} \frac{5}{36} \frac{4}{36} \frac{3}{36} \frac{2}{36} \frac{1}{36}$$

$$F(x) = \frac{1}{36}, \frac{3}{36}, \frac{6}{36}, \frac{10}{36}, \frac{15}{36}, \frac{21}{36}, \frac{28}{36}, \frac{32}{36}, \frac{36}{36} = 1$$

Hint: $X = 2$ apears only once at $(1, 1)$ while as $X = 3$ apears at $(1, 2)$ and $(2, 1)$ twice and soon.

Example: Countably infinte sample space. In tossing a coin X is number of trial till getting head. $X = \{1, 2, 3, 4..........$ if head appears
$X = \{0$ otherwise

Solution: Since events or trials are independent. getting head in any trial is independent of other trial, we can have cases as

$$P(X = 1) = P(H) = \frac{1}{2}$$

$$P(X = 2) = P(TH) = \frac{1}{2}.\frac{1}{2} = \frac{1}{2^2}$$

$$P(X = 3) = P(TTH) = \frac{1}{2}.\frac{1}{2}.\frac{1}{2} = \frac{1}{2^3}$$

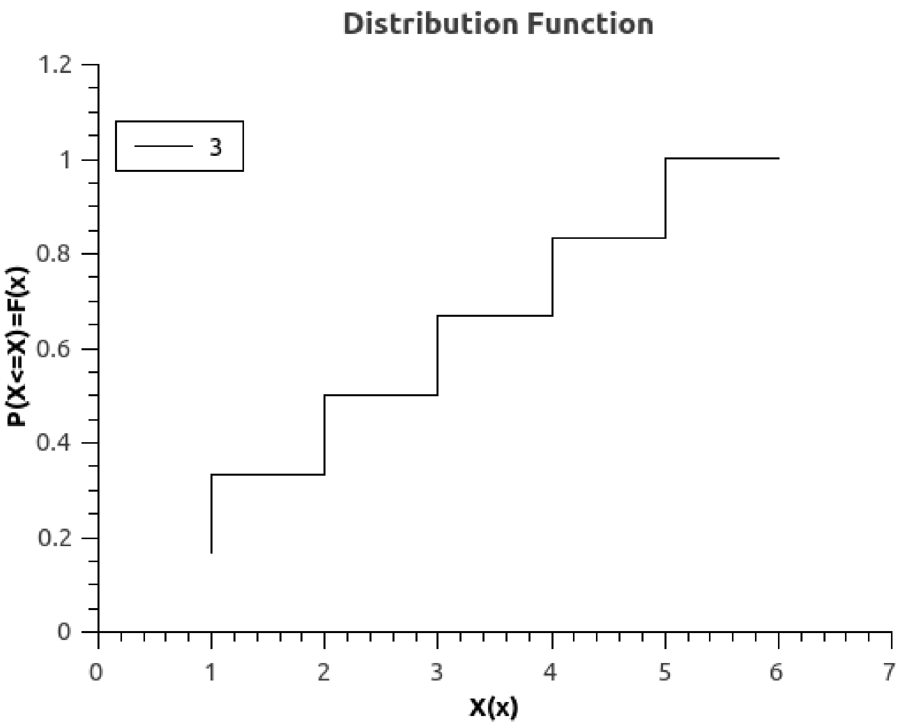

Figure 3: Probability Distribution Funciton

. . . .

$$P(X=n) = P(TTTTTT....H_n) = \frac{1}{2}\frac{1}{2}.....\frac{1}{2} = \frac{1}{2^n}$$

We know total Probability of all events or Distribution fucntion will be just 1 let us see

$$F(x) = \sum_{i=1}^{n} P_i = P(H) + P(TH) + \cdots + P(TTTT....H_n)$$

$$F(x) = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n}$$

$$F(x) = \frac{1}{2}\left(\frac{1-(\frac{1}{2})^{n-1}}{1-\frac{1}{2}}\right)$$

As $n \to \infty$ we have

$$\frac{1}{2}\left(\frac{1-0}{1-\frac{1}{2}}\right) = 1$$

s

In case of a continous random variable the probability is defined over a range and not at a fixed value of random variable like discrete one. Here probability is measured by an area under a curve defined by a function. Obviously area at single point is zero and so is the probability. For example the continous funciton we often deal with a wave function for solving a particle in a box problem. we are aware that particle is most probable at center in $n = 1$ state.

Here $F(x)$ is a continous and non decreasing behaves as

$$F(x) = 0, \qquad -\infty \leq x < x_1$$

$$F(x) = 0 + f(x_1), \qquad x_1 \leq x < x_2$$

$$F(x) = f(x_1) + f(x_2), \qquad x_2 \leq x < x_3$$

$$F(x) = f(x_1) + f(x_2) + f(x_3), \qquad x_3 \leq x < x_4$$

$$F(x) = f(x_1) + f(x_2) + \cdots + f(x_n), \qquad x_n \leq x < \infty$$

$$\lim_{x\to+\infty} F(x) = 1$$

For continous random variable we can write

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x')dx'$$

where
i)

$$f(x) \geq 0$$

ii)

$$\int_{-\infty}^{+\infty} f(x)dx = 1$$

Example: Let

$$f(x) = 0.75(1 - x^2), -1 \leq x \leq 1$$

otherwise

$$f(x) = 0$$

Find Distribution funciton $F(x)$ and find Probability $P(-\frac{1}{2} \leq X \leq \frac{1}{2}$ and $P(-\frac{1}{4} \leq X \leq 2$ and find x such that $P(X \leq x) = 0.95$ we know

$$F(x) = \int_{-\infty}^{+\infty} f(u)du = 1$$

$$= \int_{-\infty}^{-1} f(u)du + \int_{-1}^{+1} f(u)du + \int_{1}^{+\infty} f(u)du + 0 = 1$$

$$= 0.5 + 0.75x - 0.25x^3, -1 < x \leq 1$$

for $x = 1$

$$F(x) = 1$$

Similarly

$$P(-\frac{1}{2} \leq X \leq \frac{1}{2} = 0.6875 = 68\%$$

And

$$P(\frac{1}{4} \leq X \leq 2 = 31.64\%$$

And

$$P(X \leq x) = P(-1) = 0.95$$

gives $x = 0.73$

[Comment]: We have seen variables discrete and continous. Associated to these variables is a function called as probability function. for number of values of variable finite or infinite we define it in the form of Distribution. The famous Distribution you may be aware of are Bionomial, Poissoon, Normal and Gamme Distribution. Our next lecture is all about the Distributions.