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Probability Distributions..cont...

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0.1 Poisson Distribution (discret dist.)

In this distribution again success and failure is involved like Binomial, However we know only about happings and not about what does not happen. For example we know on a average how many cars enter per unit time or how many photons are detected(not missed), how many accidens occur. From this information we can estimate the Probability of any desirable event. The Probability here is proportional to size of sample say ,region,time volume or number. We know Binomial Distribution $B(n, x)$ concerned with failure and success with Probability of success as p and of failure as $(1 - p)$ having mean np and varaincenpq is following a Probability Distribution as

$$f(x) = B(n, x) = \sum_{x=0}^{n} \frac{n!}{(n-x)!x!} p^x q^{n-x}$$

A special case when $n = 1$, it is following a Distribution known as Bernouli's Distribution as

$$f(x) = B(1, x) = \sum_{x=0}^{1} \frac{1!}{(1-x)!x!} p^x q^{1-x}$$

with mean $np = 1.\frac{1}{2}$ and varaince $npq = 1.\frac{1}{2}\frac{1}{2} = \frac{1}{4}$

When n the number of trials is very very large or when $n \to \infty$ say number of photons detected by a dector in some given time or number of births or deaths taking place in some town or millions of coins tossed together or coin tossed a million times, in this situation the Probability per unit event approaches to zero as for one coin it $\frac{1}{2}$ for two coins having four possible events ,each with Probability as $\frac{1}{2^2}$ and for n it is $\frac{1}{2^n} \to 0$ for large n. However the mean say $\lambda = np$ is taking some finite value due to large n. The Distribution which our randome variable X follows here well is known to be Poisson Distribution with mean λ.

In this Distribution we do not know the source as it goes to infinity n, the given information with us is just average. for example per unit time these many cars enter the parking slot, per unit day these many accidens on average take place, per unit time these many photons on an average are detected and soon. We know accidens rate but not out of how many cars. we know average birhts and not the population behind or souce.

we can develop this Poisson Distribution independently as well as from Binomial as $n \to$

0.2 Poisson Distribution from Binomial Distribution

We know Binomial Distribution as

$$f(x) = P(X = x) = \frac{n!}{(n-x)!x!} p^x q^{n-x}$$

$$= \frac{n(n-1)(n-2)\cdots(n-(x-1))(n-x)!}{(n-x)!x!} p^x (1-p)^{n-x}$$

$$= \frac{n^x}{x!} p^x (1-p)^{n-x}$$

$$= \frac{(np)^x}{x!} (1-p)^{n-x}$$

As $n \to \infty$, $\quad p \to 0$; $\quad n - x \to n$

$$f(x) = \frac{(np)^x}{x!}(1-p)^n == \frac{(np)^x}{x!}[1 - np + \frac{n(n-1)}{2!}p^2 - \frac{n(n-1)(n-2)}{3!}p^3 + ...]$$

$$f(x) = \frac{(np)^x}{x!} e^{-np}$$

Therefore Poisson Distribution is after putting $np = \lambda$ which we see it to be mean of the Distribution is

$$P(X = x) = f(x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

Or we write a randam varaible follows a Poisson with mean λ as

$$P(x; \lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$$

0.3 Expectation Value of Poisson Distribution E(X):

$$E(X) = \sum_{x=0}^{n} x P(x; \lambda) = \sum_{x=0}^{n} \frac{x \lambda^x e^{-\lambda}}{x!}$$

$$= \sum_{x=1}^{n} \frac{x \lambda^x e^{-\lambda}}{(x-1)!}$$

on substituting

$$(x-1) = y, \quad (n-1) = m$$

we get

$$E(X) = e^{-\lambda} \sum_{y=0}^{m} \frac{\lambda^{y+1}}{y!}$$

$$E(X) = e^{-\lambda}[\lambda + \frac{\lambda^2}{1!} + \frac{\lambda^3}{2!} + \frac{\lambda^4}{3!} + ...]$$

$$E(X) = \lambda e^{-\lambda}[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + ...]$$

$$E(X) = \lambda e^{-\lambda} e^{\lambda} = \lambda$$

so the expectation vaule of Poisson Distribution is λ

0.4 Varaince $\sigma^2(X)$:

$$\sigma^2 = E(X-\mu)^2 = \sum_{x=0}^{n}(x-\mu)^2 P(x;\lambda) = \sum_{x=0}^{n}(x-\mu)^2 \frac{\lambda^x e^{-\lambda}}{x!}$$

we know

$$\sigma^2 = E(X(X-1)) + E(X) - [E(X)]^2$$

expectation is already known, We need to just know $E(X(X-1))$ and we are done

$$E(X(X-1)) = \sum_{x=0}^{n}(x(x-1)\frac{\lambda^x e^{-\lambda}}{x!}$$

$$E(X(X-1)) = \sum_{x=2}^{n}\frac{\lambda^x e^{-\lambda}}{(x-2)!}$$

on substituting $(x-2) = y$ and play of mathematics we get

$$E(X(X-1)) = \sum_{y=0}^{m}\frac{\lambda^{y+2} e^{-\lambda}}{y!}$$

$$E(X(X-1)) = \lambda^2 \sum_{y=0}^{m}\frac{e^{-\lambda}\lambda^y}{y!} = \lambda^2.1 = \lambda^2$$

Therefore

$$\sigma^2 = E(X(X-1)) + E(X) - [E(X)]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

Hence we see expectation as well as the varaince for Poisson Distribution is same as λ

0.5 Poisson Distribution (Alternative Method):

Here events take place independently. Occurance of one event has no influence on the Occurance of other.Remember n is unknown here compared to Binomial dist.Only average happings is known.

we will take three situations for deriving Poisson Distribution as summarised here

Interval	**Events considered**	**Probability Possibility**
δt	maximum a single event	$P(1;\delta t) = \lambda\delta t$
$t+\delta t$	zero event	$P(0;\delta t).P(0;=t)$
$t+\delta t$	x events	$P(x-1;t).P(1;=\delta t) + P(x;t).P(0;=\delta t)$

Probability of single event taking place in time δt is

$$P(1, \delta t) \;\; \alpha \;\; \delta t$$

$$P(1, \delta t) = \lambda \delta t$$

λ a proportionality constant.

And a Probability that no event (zero event) takes place in δt

$$P(0, \delta t) = \lambda \delta t$$

Therefore total Probability in time δt

$$P(1, \delta t) + P(0, \delta t) = 1$$

OR

$$P(0, \delta t) = 1 - P(1, \delta t) = 1 - \lambda \delta t$$

$$P(0, \delta t) = 1 - \lambda \delta t$$

Let us consider the size or time window as $t + \delta t$ and assume not a single event takes place in this duration , we can write keeping in view the independent events

$$P(0, t + \delta t) = P(0, t).P(0, \delta t) = P(0, t)[1 - \lambda \delta t]$$

$$\frac{P(0, t + \delta t) - P(0, t)}{\delta t} = -\lambda P(0, t)$$

$$\frac{dP(0, t)}{P(0, t)} = -\lambda dt$$

$$\ln P(0, t) = -\lambda t + c$$

at $t = 0$ no event take place is for sure so

$$P(0, t = 0) = 1$$

Therefore

$$c = 1$$

Hence

$$P(0, t) = e^{-} \lambda t$$

Now consider the third situation in time $t + \delta t$ let x number of events take place, it can happen (keeping in view only single event takes place in δt as either x even occurs in t and 0 event in δt

OR

$x - 1$ even occurs in t and 1 event in δt

Therefore we have

$$P(x; t + \delta t) = P(x; t)P(0, \delta t) + P((x - 1; t).P(1, \delta t)$$

$$P(x; t + \delta t) = P(x; t)(1 - \lambda\delta t) + P((x - 1; t).\lambda\delta t$$

$$\frac{P(x; t + \delta t) - P(x; t)}{\delta t} + \lambda P((x; t) = .\lambda(Px - 1; t)$$

$$\frac{dP(x; t)}{dt} + +\lambda P((x; t) = \lambda(Px - 1; t)$$

Multiply on both the sides by $e^{\lambda t}$

$$\frac{d}{dt}[e^{\lambda t}P(x; t)] = e^{\lambda t}P(x - 1, t)$$

for $x = 1$ the equation becomes as

$$\frac{d}{dt}[e^{\lambda t}P(1; t)] = e^{\lambda t}P(0, t)$$

we know

$$P(0; t) = e^{-\lambda t}$$

$$\frac{d}{dt}[[e^{\lambda t}P(1; t)] = \lambda$$

Integrate we have

$$e^{\lambda t}P(1; t)] = \int \lambda dt = \lambda t + c$$

at $t = 0$ no event takes place so $c = 0$ Therefore

$$P(1; t) = \lambda te^{-\lambda t} = \left(\frac{\lambda t}{1!}\right)^1 e^{-\lambda t}$$

Similarly we have for $x = 2$

$$P(2; t) = \left(\frac{\lambda t}{2!}\right)^2 e^{-\lambda t}$$

and for any x event we have

$$P(x;t) = \left(\frac{\lambda t}{x!}\right)^x e^{-\lambda t}$$

Put this $\lambda t = \mu$ we have

$$P(x;t) = \frac{\mu^x e^{-\mu}}{x!}$$

This is the desired Poisson Distribution with mean μ and varaince μ we see the total Probability or distribution function is just one as

$$\sum_{x=0}^{\infty} P(x;\mu) = \sum_{x=0}^{\infty} \frac{\mu^x e^{-\mu}}{x!} = e^{-\mu} \sum_{x=0}^{\infty} \frac{\mu^x}{x!} = e^{-\mu} e^{\mu} = 1$$

Example: If on an average 2 cars enter per minute. What is the Probability that during any given minute 4 or more cars enter.

Solution:

$$np = \lambda = 2$$

$$P(x;\lambda = 2) = \frac{\lambda^x e^{\lambda}}{x!} = \frac{2^x e^2}{x!}$$

Now using complemnet

$$P(x \geq 4; 2) = 1 - P(x < 4; 2) = 1 - [P(x = 0; 2) + P(x = 1; 2) + P(x = 2; 2) + P(x = 3; 2)]$$

$$= 1 - e^{-2}\left[\frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!}\right] = 14.3\%$$

Example: The average number of accidents at a level-crossing every year is 5. Calculate the probability that there are exactly 3 accidents there this year.

Solution: Here, $\lambda = 5$ and you need to Calculate $P(x = 3, 5)$. Answer: 14%

0.6 Problems on Poisson Distribution (Home Work)

1. A radioactive source emits 4 particles on average during a five-second period.
a) Calculate the probability that it emits 3 particles during a 5-second period.
b) Calculate the probability that it emits at least one particle during a 5 second period.
c) During a ten-second period, what is the probability that 6 particles are emitted?

2. The number of typing mistakes made by a secretary has a Poisson distribution. The mistakes are made independently at an average rate of 1.65 per page. Find the probability that a three-page letter contains no mistakes.

3. A 5-litre bucket of water is taken from a swamp. The water contains 75 mosquito larvae. A 200 mL flask of water is taken form the bucket for further analysis. What is

a) the expected number of larvae in the flask? b) the probability that the flask contains at least one mosquito lava?

4. If the light bulbs in a house fail according to a Poisson law, and over the last 15 weeks there have been 5 failures, find the probability that there will not be more than one failure next week.