Contents

Probability

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1 Probability

- There are two types of process in nature. One where future is surely pridcited called as *Deterministic Process* and the one where future is not surely predicted called as *Probabilistic Process.*
- Probability earlier was considered to be just a qualitative statement but with the advance in the subject brought it a precisely quatitative meaning ranging from 0 to 100 percent.
- The concept of Probability is used almost in every aspect of day to day matte (think of various examples, car accident, death rate, birht rate, win or lost, prob. of head and tail in coin toss, for single toss head and tail ie equally probable but we may get different numbers as well as it is probable and not sure.. for for 10 coin toss not necassary getting each 5times).
- Probability is not the answer but a guide and tends to be answer when sample space is infinte.For example out 10 coin toss you may not get 50% head and tail but out of millions coin toss you most probabbly will get 50% head and 50% tail.

1.1 Deterministic Process

Examples are as

1. Ohm's law, Voltage drop is always proportional to Current $V = IR$ or $V \propto I$
2. Gas law $PV = nRT$ or $P \propto V^{-1}$
3. Newtons laws of motion

1.2 Probabilistic Process

These are random processes in nature where future is not surely predicted as above . We call them unpredictable processes but a Probabilistic value can be assigned predicted in advance .

Probability is frequently used for business, economics, science or day to day life.

1.3 Some Basic Terms

Randome Exaperiment We mean an experiment having multiple possible outcomes. And one does not know in advance which outcome is going to occur.We can just pridict the chances of outcome but not necassaryly to get that. for example tossing a coin, both the head and tail have 50% chance. But we may get always head and never tial is a possibility.

Throwing a die, tossing a coin, Throwing a card, in a bad containing red and green balls and choosing randomly one ball can be red or can be green etc are the examples of random experiment.

Sample Space (S) : All possible outcome of a random experiment is know as sample space.
For tossing a single coin
$S = \{H, T\}$

For tossing twocoins

$S = \{HH, HT, TH, TT\}$

We can write in number as noting down the number of head occured $S = \{2, 1, 0H, T\}$

Similary Throwing a die

$S = \{1, 2, 3, 4, 5, 6\}$

Throwing two dices together we get 36 possible outcomes

$S = \{11, 12, 13, 14, 15, 16, 21, 22, 23, 24, 25, 26, 31, 32, 33, 34, 35, 36, 41, 42, 43, 44, 45, 46, 51, 52, 53, 54, 55, 56, 61, 62, 63, 64, 65, 66\}$

Similary throwing two dice , the sum is 4 we get 5 possible outcomes

$14, 23, 32, 41$

Similary throwing two dice , look for the sum is $4, 7, 9, 12,$

Event (E) : Subset of the sample space is known as event. for two dice throw, we have 36 possible outcomes and each outcome is an event. Tossing a coin we have two events Head and Tail. We can also define a Subset of a sample space as event.

Let A is an event of odd and B is an event of even in a single dice, we have

$A = \{1, 3, 5\}$

$B = \{2, 4, 6\}$

Similary we can define event say C as sum of two dices is 7

$C = \{16, 25, 34, 43, 52, 61\}$

Mutually Exclusive Events (ME) : Two events A and B are said to be Mutually Exclusive Events if occurance of A prevents the occurance of B. The common of A and B is 0 called as null set.

$A \cap B = \phi$

Let A event of more heads and B is an event of head on last throw

Mutually Exclusive

Turning left and turning right are ME (can't do both) Tossing a coin: Heads and Tails are Mutually Exclusive Cards: Kings and Aces are Mutually Exclusive

What is not Mutually Exclusive:

Turning left and scratching your head can happen at the same time Kings and Hearts, because we can have a King of Hearts!

Think of thrice we get
$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$
$A = \{HHH, HHT, HTH, THH, \}$
$B = \{HHH, HTH, THH, TTH\}$

Idependent Events: Two events are said to be independent if occurance of the one does not affact the occurance of other. for example throwing two dices, getting a on one dice is independent of getting anything on other dice.
While as Mutually Exclusive events are the most dependent kind of events.

if event A and B are the independent events then we have

$$P(AB) = P(A).P(B)$$

Complement Events : The Complement $\bar{A}$ of A consist of all the points (events) of sample space S not in A, that is everything of S but A.

$$A \cup \bar{A}) = S$$

$$P(A \cup \bar{A}) = P(S) = 1$$

and

$$P(A \cap \bar{A}) = 0$$

for example tossing a coin if $A = H$ then $\bar{A} = T$

$$A \cup \bar{A}) = \{H, T\} = S$$

A and $\bar{A}$ are the disjoint sets or mutually Exclusive. we see

$$P(A \cup \bar{A}) = P(A) + P() = P(S) = 1$$

OR

$$P(A) + P() = P(S)$$

$$P(\bar{A}) = P(S) = 1 - P(A)$$

This is a useful expressions.

Probability P: The probability of an event is defined as the number of favourable cases n per number of all possible cases N.

$$P(event) = \frac{n}{N}$$

let event A and B contain n_A and n_B cases respectively then we can define

$$P(A) = \frac{n_A}{N}$$

$$P(B) = \frac{n_B}{N}$$

For example in the above cases

$$P(A) = \frac{4}{8}$$

similarly

$$P(B) = \frac{4}{8}$$

from above we see

$$(A \cap B) = \{HHH, HTH, THH\}$$

$$P(A \cap B) = \frac{3}{8}$$

we also see

$$P(A \cap B) \neq P(A).P(B)$$

we will see to define such a case is dependent and not the independent one. or we can say A and B are not Mutually Exclusive Events.

New Events : We can build up new events subsets or sets from the old ones as

$(A \cup B)$: consist of events of A, B and common of A and B
$(A \cap B)$: consist of only common events of A, B
(A/B) : consist of events of A minus events of B . That is consist of events of A not of B
$O/$ (empty set) for the events which does not contain any outcome.
....

Axioms (self evident): For any event E we can see or define

1. $P(E) \geq 0$

2. $P(S) = 1$

3. $P(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} P(E_i)$ = called as axiom of additivity.

1.4 Examples

1. Toss a coin 2 times, what is the probability
a) at most one head b) at least one head
c) just one head

2. In a die throw what is the probaibility of getting a) more than 4
3. In a two die throw what is the probablitiy of getting a) total 7 b) of gettting double c) getting total of 10 or 11

independent Events

3. For tossing a coin we know

$$P(H) = P(T) = 0.5$$

upon tossing a coin 5 times. each time a coin is tossed is independent events (think). so

$$P(ABCDE) = P(A)P(B)P(C)P(D)P(E)$$

or

$$P(HTHTH) = P(HHHHH) = P(TTTTT) = 0.5{*}0.5{*}0.5{*}0.5{*}0.5 = (0.5)^5 = 0.03$$

Two Mutually Exclusive events have zero joint probability . for a coin toss Example:

$$P(AB) = 0 = P(HT) = P(TH) = 0$$

Example: In a two dice throw. Find the probability that the sum of two numbers that appear is not 6 using Complement concept. let A is representing sum is not 6 even then $\bar{A}$ represent everything but 6

$$P(A) = P(S) = 1 - P(\bar{A}) = 1 - \frac{5}{36}$$

for reference see table below.

$\rightarrow$	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

1.5 Home work: Look at Venn diagrams

1.6 Conditional Probability:

If a bag contains 7 red balls and 3 green balls. The probability of having a red ball in hand is going to be different that the probability of having green ball. Both of them is going to be different than simply having a ball in hand. With this we can define for the probability of an event B with a condition that an event A occurs as conditional probability of B given A as $P(B/A)$.

In such situation condition brings forth a new sample space (reduced or resized sample space). Here we define

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

1.6.1 Examples:

1. The probability of having a king in a pack of 52 cards is $\frac{4}{52}$ while as having a red king (condition) is just $\frac{2}{52}$

$$P(king)/P(RedKing) = \frac{2}{52}$$

In this case common points are only two red kings out of 52 so $(A \cap B) = 2$ and $P(A \cap B) = \frac{2}{52}$

1.6.2 Derivation of conditional probability

Let event B occurs and given B is from A so $B \in A$. Let N be total sample space. Let N_A be the sample space of event A. Let N_{AB} is the common of event AB therefore

$$P(A \cap B) = \frac{N_{AB}}{N} = \frac{N_{AB}}{N_A} . \frac{N_A}{N}$$

clearly

$$\frac{N_{AB}}{N_A} = P(B/A)$$

and

$$\frac{N_A}{N} = P(A)$$

therfore

$$P(A \cap B) = P(B/A).P(A)$$

OR

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

similarly we have FOR THE REVERSE situation
NOTE: if event B is ndependednt of event A then we have

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = P(B)$$

which gives us for independent cases

$$P(B).P(A) = P(A \cap B)$$

Example: Throw a fair single dice define events as $B = \{6\}$, $A = \{2, 4, 6\}$ the conditional probability $P(B/A)$ is given as

[h]

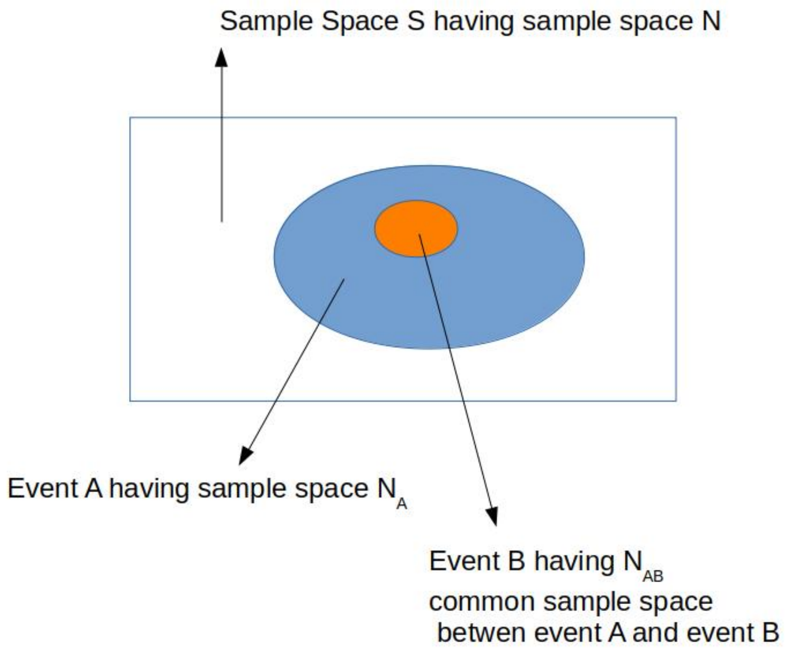

Figure 1: conditional probability

$$P(B/A) = \frac{(A \cap B)}{P(A)} = \frac{\frac{1}{6}}{\frac{3}{6}} = \frac{1}{3}$$

Note down the new or reduced sample space is just $\{2, 4, 6\}$

Example: for a fair single dice throw what is the probability if getting an event $A = \{1\}$ given that $B = \{1, 3, 5\}$ occured

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1}{3}$$

As reduced sample space is $\{1, 3, 5\}$

Example: With and without replacement.

A box conains 10 screws, 3 of which are defective. 2 screws are drawn at random. Find the probability that none of the screws are defective.
solution
a) With Replacement. Let first screw is non defective is event A
Let second screw is non defective is event B

$$P(A) = \frac{7}{10}, P(B) = \frac{7}{10}$$

s as they are two independent cases. Now the probability that both the screws are non defective $P(A \cap B)$

$$P(A \cap B) = P(A).P(B) = \frac{7}{10}\frac{7}{10} = \frac{49}{100} = 49\%$$

b) Without Replacement (conditional case)

$$P(A) = \frac{7}{10}$$

and

$$P(B/A) = \frac{6}{9} = \frac{2}{3}$$

Now Probability that both the screws are non defective

$$P(A \cap B) = \frac{7}{10}\frac{6}{9} = 47\%$$

1.7 Independent Events:

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

(1)

and

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

(2)

solving equations 1 and 2

$$P(B/A) = \frac{P(B).P(A/B)}{P(A)}$$

(3)

And when events are independent as the case is here

$$P(B/A) = P(B), P(A/B) = P(A)$$

therforefrom equaiton 3

$$P(B/A) = P(B) = \frac{P(B).P(A)}{P(A)}$$

from equaiton 1

$$P(B) = \frac{P(A \cap B)}{P(A)}$$

$$P(B).P(A) = P(A \cap B)$$

or

$$P(A \cap B) = P(B).P(A)$$

for m independent events we have

$$P(A_1 \cap A_2 \cap A_3 \cap A_4 \cdots \cap A_m) = P(A_1).P(A_2) \cdots P(A_m)$$

1.8 Additon theorem of probability:

If A and B are not disjoint events in sample space S. We have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Solution: From figure 2

$$A \cup B = A \cup (\bar{A} \cap B)$$

from the figure it is clear that A and $(\bar{A} \cap B)$ are disjoint. so Probability is

$$P(A \cup B) = P(A) + P(\bar{A} \cap B) = P(A) + P(B) - P(A \cap B)$$

OR

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Home Work: For three non mutually Exclusive events A, B, C

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

[h]

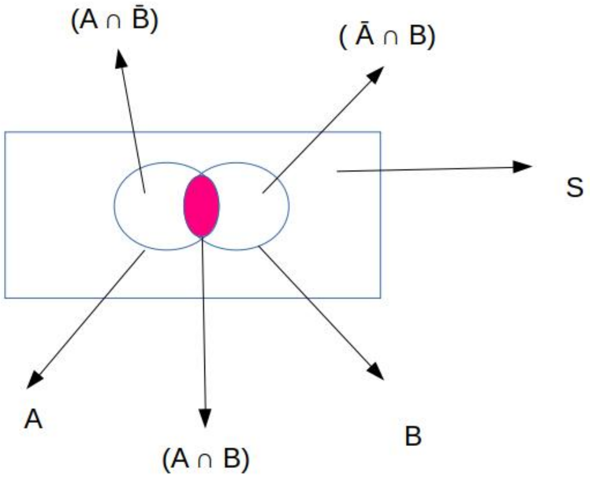

Figure 2: venn diag

1.9 Law of Total Probability:

If $S = \{A_1, A_2, A_3, A_4, A_5, \cdots, A_n\}$, and if these A_i are mutually Exclusive events, then

$$P(A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup \cdots \cup A_n) = P(A_1) + P(A_2 + \cdots + P(A_n)$$

[h]

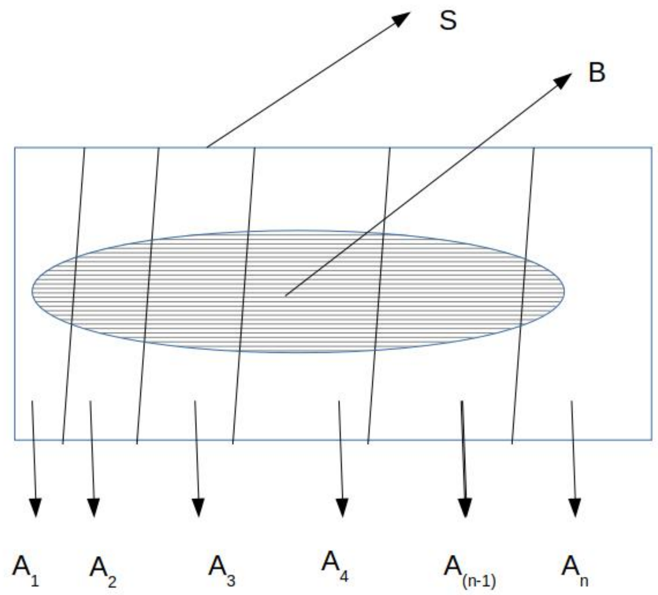

Figure 3: Law of Toal Probability

And if these A_i exaust S that is their union is S as

$$A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup \cdots \cup A_n = S$$

therefore

$$P(A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup \cdots \cup A_n) = P(S) = 1$$

Let us derive a formula for law of total probability $P(A \cap B)$

Let

$$A = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup \cdots \cup A_n$$

$$(A \cap B) = (A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup \cdots \cup A_n) \cap B$$

$$(A \cap B) = (A_1 \cap B) \cup (A_2 \cap B) \cup (A_3 \cap B) \cup (A_4 \cap B) \cup (A_5 \cap B) \cup \cdots \cup (A_n \cap B)$$

From law of addition of probabilities given as

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

we have

$$P(A \cap B) = \sum_{i=1}^{n} P(A_i \cap B)$$

(4)

for mutually exclusive events

Since A_i exaust S that is there union exhust S , therefore

$$A \cap B = S \cap B = B$$

$$P(A \cap B) = P(B) = \sum_{i=1}^{n} P(A_i \cap B)$$

OR

$$P(B) = \sum_{i=1}^{n} P(A_i \cap B)$$

OR

$$P(B) = \sum_{i=1}^{n} P(B/A_i).P(A_i)$$

which is the law of total probability.

Here we are interested in finding the probability of an event B, but we don't know how to find $P(B)$ directly. Instead, we know the conditional probability of B given some events A_i, where the A_i form a partition of the sample space. Thus, we will be able to find $P(BA)$ using the law of total probability given above.

Example:

Factory X supply lamps which work $> 5000hrs$ in 99% cases while facotry Y supply lamps which work $> 95\%$ cases. Facotry X produce 60% of supply and facotry Y produce 40%. What is the probability $P(lamp \geq 5000hrs)$

Solution:

$$P(A > 5000hrs) = P(A/B_X).P(B_X) + P(A/B_Y).P(B_Y)$$

$$P(A > 5000hrs) = 99\%.60\% + 95\%.40\% \ = 97.4\%$$

thus each lamp purchased has a chance 97.4% to work $> 5000hrs$

Example: I have three bags that each contain 100 marbles:Bag 1 has 75 red and 25 blue marbles;
Bag 2 has 60 red and 40 blue marbles; Bag 3 has 45 red and 55 blue marbles.I choose one of the bags at random and then pick a marble from the chosen bag, also at random. What is the probability that the chosen marble is red?

Solution Hint: Ball has to come from one of the bag which is equallyprobably with probability $P(B_i) = \frac{1}{3}$.

and ball from each bag varies with probability as for first $P(R/B_1) = 0.75$ then calculate total probability for selecting a red ball as per the law of total probability.

$$P(R) = \sum_{i=1}^{3} P(R/B_i).P(R_i)$$

Example : A bucket contains 6 red 4 green balls. Two balls are drawn without replacement. Find

$$P(R_1)$$

,

$$P(R_2/R_1)$$

,

$$P(R_2)$$

Are the two events so drawn as R_1 and R_2 Independent. Justify.

Solution:a)

$$P(R_1) = \frac{6}{10}$$

b) (i) Directly

$$P(R_2/R_1) = \frac{5}{9}$$

(ii) this can be done also as per the conditional probability

$$P(R_2/R_1) = \frac{P(R_1 \cap R_2)}{P(R_1)} = \frac{P(R_1).P(R_2)}{P(R_1)} = \frac{5}{9}$$

c)

$$P(R_2) = P(R_2/R_1).P(R_1) + P(R_2/G_1).P(G_1) = \frac{5}{9}.\frac{3}{5} + \frac{6}{9}.\frac{4}{10}$$

Home Work Prove Bayes theorem